ON THE SPATIAL PROBLEM OF WAVES ON THE SURFACE OF A VISCOUS FLUID OF INFINITE DEPTH

(K PROSTRANSTVENNOI ZADACHE O VOLNAKH NA POVERKHNOSTI VIAZKOI ZHIDKOSTI BESKONECHNOI GLUBINY)

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A.K.NIKITIN and S.A.PODREZOV
(Rostov-on-Don)

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The linear spatial problem of gravitational waves on the surface of a viscous incompressible fluid of infinite depth which are generated from a state of rest under the action of a surface pressure and an initial disturbance of the free surface is considered. By successive application of a multiple Fourier transform with respect to the coordinates and a Laplace transform with respect to time, a solution of the problem is obtained in closed form. The solution of the axisymmetric problem is obtained as a particular case. Asymptotic formulas which make it possible to easily solve specific problems are then obtained.

1. The equation of motion, boundary and initial conditions. Let us locate the origin of a rectangular Cartesian coordinate system on the free surface of a fluid in a state of equilibrium. Let us direct the z-axis vertically upwards and the x and y axes in the horizontal plane. We shall assume that at the initial moment of time the velocities of the fluid particles are equal to zero and at that the form of the free surface is given by $z = \zeta(x, y, 0) = \zeta_0(x, y)$; the pressure on the free surface is given by $p = p_0(x, y, t)$ and the frictional stress vanishes there. Under the usual assumptions of linear theory, neglecting the nonlinear inertial terms, we obtain the equations of motion

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta v_x \qquad (xyz), \qquad p = p_1 + \rho g z \tag{1.1}$$

and the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{1.2}$$

where p_1 is the hydrodynamic pressure. In addition, we shall have the following boundary conditions:

$$p_{nn} = -p_0, \qquad p_{n\tau_1} = 0, \qquad p_{n\tau_2} = 0 \qquad \text{on } z = \zeta(x, y, t) \tag{1.3}$$

$$v_x = v_y = v_z = 0 \quad \text{for } z = -\infty, \qquad v_x = v_y = v_z = 0 \quad \text{for } t = 0 \tag{1.4}$$

Here $z = \zeta(x, y, t)$ is the equation of the free surface.

We shall assume the motion to be slow, the wave amplitude to be small and the waves to be gently sloping. It is then possible to consider that the normal to the free surface deviates only slightly from the vertical and to set

$$p_{n\tau_1} \approx p_{zx} = 0, \quad p_{n\tau_2} \approx p_{zy} = 0 \quad \text{for } z = \zeta(x, y, t) \tag{1.5}$$

$$p_{nn} \approx p_{zz} = -p_1 + 2\mu \frac{\partial v_z}{\partial z} = -p + \rho g \zeta + 2\mu \frac{\partial v_z}{\partial z} = -p_0(x, y, t)$$

Let us apply conditions (1.5) ti the undisturbed surface $z = 0$. Taking into consideration that for small oscillations

$$\frac{\partial \zeta}{\partial t} = (v_z)_{z=0}, \qquad \zeta = \zeta_0(x, y) + \int_0^t (v_z)_{z=0} dt \tag{1.6}$$

we obtain from (1.5)

$$\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} = 0, \qquad \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} = 0 \qquad \text{for } z = 0$$

$$p - \rho g \int_0^t v_z dt - 2\mu \frac{\partial v_z}{\partial z} = p_0(x, y, t) + \rho g \zeta_0(x, y) \tag{1.7}$$

2. Application of the Fourier and Laplace transforms. To solve the problem we shall apply the multiple Fourier transform [1] with respect to the variables x and y and then the Laplace transform [2] with respect to time t.

Let us multiply the equations (1.1), (1.2) and the conditions (1.4), (1.6) and (1.7) by

$$\tfrac{1}{2}\pi^{-1} \exp\,[i(\xi x + \eta y)]\, dxdy$$

and integrate the results with respect to x and y from $-\infty$ to $+\infty$.

Assuming that the quantities p, v_x, v_y, v_z, $\partial v_x / \partial x$, $\partial v_z / \partial x$, $\partial v_x / \partial y$, $\partial v_y / \partial y$, $\partial v_y / \partial x$, $\partial v_z / \partial y$ vanish for $|x| \to \infty$, $|y| \to \infty$, we have

$$\frac{\partial X}{\partial t} = \frac{i\xi}{\rho} P + \nu\left(\frac{\partial^2 X}{\partial z^2} - \xi^2 X - \eta^2 X\right)$$

$$\frac{\partial Y}{\partial t} = \frac{i\eta}{\rho} P + \nu\left(\frac{\partial^2 Y}{\partial z^2} - \xi^2 Y - \eta^2 Y\right)$$

$$\frac{\partial Z}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 Z}{\partial z^2} - \xi^2 Z - \eta^2 Z\right) \tag{2.1}$$

$$\frac{\partial Z}{\partial z} - i(\xi X + \eta Y) = 0, \qquad \frac{\partial X}{\partial z} - i\xi Z = 0, \qquad \frac{\partial Y}{\partial z} - i\eta Z = 0$$

$$P - \rho g \int_0^t Z dt - 2\mu \frac{\partial Z}{\partial z} = P_0 + \rho g H_0 \quad \text{for } z = 0 \tag{2.2}$$

$$X = Y = Z = 0 \quad \text{for } z = -\infty, \qquad X = Y = Z = 0 \quad \text{for } t = 0 \tag{2.3}$$

Here the Fourier transforms of the functions v_x, v_y, v_z, p, ζ, ζ_0 and p_0 are denoted, respectively, by X, Y, Z, P, H, H_0 and P_0. It is assumed that the Fourier transform is applicable to the quantities under consideration. Taking into account that the fluid motion arises from a state of rest and is induced by the surface pressure $p_0(x, y, t)$ and the initial disturbance of the free surface $\zeta_0(x, y)$, it can be considered that the indicated conditions will be satisfied if Fourier transforms are applicable to the functions $p_0(x, y, t)$ and $\zeta_0(x, y)$. (If $p \to a$ = const for $|x| \to \infty$ and $|y| \to \infty$, it is then possible to consider the quantities $p - a$ and $p_0 - a$ in place of p and p_0, respectively, in the equations and the boundary conditions).

Let us now apply the Laplace transform with respect to t to Equations (2.1) and the boundary conditions (2.2) and (2.3). Taking into account the initial conditions (2.3), we obtain

$$\begin{gathered} \frac{d^2X^\circ}{dz^2} - \left(\frac{s}{\nu} + \xi^2 + \eta^2\right) X^\circ + \frac{i\xi}{\mu} P^\circ = 0 \\ \frac{d^2Y^\circ}{dz^2} - \left(\frac{s}{\nu} + \xi^2 + \eta^2\right) Y^\circ + \frac{i\eta}{\mu} P^\circ = 0 \\ \frac{d^2Z^0}{dz^2} - \left(\frac{s}{\nu} + \xi^2 + \eta^2\right) Z^\circ - \frac{1}{\mu}\frac{dP^\circ}{dz} = 0 \\ i\xi X^\circ + i\eta Y^\circ - \frac{dZ^\circ}{dz} = 0 \end{gathered} \tag{2.4}$$

$$\begin{gathered} P^\circ - \frac{\rho g}{s} Z^\circ - 2\mu \frac{dZ^\circ}{dz} = P_0{}^\circ + \rho g H_0 \\ \frac{dX}{dz} - i\xi Z^\circ = 0, \quad \frac{dY^\circ}{dz} - i\eta Z^\circ = 0 \end{gathered} \quad \text{for } z = 0 \tag{2.5}$$

$$\begin{gathered} X^\circ = Y^\circ = Z^\circ = 0 \quad \text{for } z = -\infty \\ H = H_0 + s^{-1} (Z^\circ)_{z=0} \qquad \left(X^\circ(s) = s \int_0^t X(t) e^{-st} dt\right) \end{gathered} \tag{2.6}$$

Here s is the transform parameter.

Eliminating X°, Y° and P° from Equations (2.4) and the boundary conditions (2.5), we obtain Equation

$$\frac{d^4 Z^\circ}{dz^4} - \left(\frac{s}{\nu} + 2r^2\right)\frac{d^2Z^\circ}{dz^2} + r^2\left(\frac{s}{\nu} + r^2\right) Z^\circ = 0 \tag{2.7}$$

$$(r^2 = \xi^2 + \eta^2)$$

for Z°, and the boundary conditions

$$\frac{d^2Z^\circ}{dz^2} + z^2Z^\circ = 0 \quad \text{for } z = 0$$

$$\frac{d^3Z^\circ}{dZ^3} - \left(\frac{s}{\nu} + 3r^2\right)\frac{dZ^\circ}{dz} - \frac{\rho g r^2}{s\mu} Z^\circ = \frac{r^2}{\mu}(P^\circ_0 + \rho g H_0) \tag{2.8}$$

$$Z^\circ = \frac{dZ^\circ}{dz} = 0 \quad \text{for } z = -\infty \tag{2.9}$$

The solution of Equation (2.7) which satisfies conditions (2.8) and (2.9) has the form

$$Z^\circ = r(\rho g H_0 + P_0^\circ)\,[\mu\Phi(r, s)]^{-1}\,[2r^2 \exp(z\sqrt{r^2 + s/\nu}) - (2r^2 - s/\nu)\exp(rz)] \tag{2.10}$$

$$\Phi(r, s) = (2r^2 + s/\nu)^2 - 4r^3(r^2 + s/\nu)^{1/2} + gr\nu^{-2}$$

Using the relations (2.4) and (2.5) for Z°, X°, Y°, H° and P°, we find

$$X^\circ = -\frac{\xi i(\rho g H_0 + P_0^\circ)}{\mu\Phi(r, s)}\left[2r\left(r^2 + \frac{s}{\nu}\right)^{1/2} e^{(z\sqrt{r^2+s/\nu})} - \left(2r^2 + \frac{s}{\nu}\right)e^{(rz)}\right]$$

$$Y^\circ = -\frac{i\eta(\rho g H_0 + P_0^\circ)}{\mu\Phi(r, s)}\left[2r\left(r^2 + \frac{s}{\nu}\right)^{1/2} e^{(z\sqrt{r^2+s/\nu})} - \left(2r^2 + \frac{s}{\nu}\right)e^{(rz)}\right]$$

$$H^\circ = H_0 - \frac{r(\rho g H_0 + P_0^\circ)}{\mu\nu\Phi(r; s)} \tag{2.11}$$

3. Determination of the originals. Let us set $u = (r^2 + s/\nu)^{\frac{1}{2}}$ in Expressions (2.10) and (2.11). Next we shall use a second expansion theorem. In accordance with [3] we find the originals of the transforms

$$e^{uz} \div \frac{1}{2}\left[e^{-r|z|}\operatorname{erfc}\left(\frac{|z|}{2\sqrt{\nu t}} - r\sqrt{\nu t}\right) + e^{-r|z|}\operatorname{erfc}\left(\frac{|z|}{2\sqrt{\nu t}} + r\sqrt{\nu t}\right)\right] = f_1(r, z, t) \tag{3.1}$$

$$\frac{u}{u-u_k}e^{uz} \div \frac{r}{2}\left[\frac{e^{-r|z|}}{r-u_k}\operatorname{erfc}\left(\frac{|z|}{2\sqrt{\nu t}} - r\sqrt{\nu t}\right) + \operatorname{erfc}\left(\frac{|z|}{2\sqrt{\nu t}} + r\sqrt{\nu t}\right)\right] \times$$
$$\times \frac{e^{-r|z|}}{r+u_k} + \frac{u_k^2\exp[-u_k|z| + (u_k^2 - r^2)\nu t]}{u_k^2 - r^2}\operatorname{erfc}\left(\frac{|z|}{2\sqrt{\nu t}} - u_k\sqrt{\nu t}\right) = f_2^k(r, z, t)$$

$$u(u-u_k)^{-1} \div -ru_k\operatorname{efr}(r\sqrt{\nu t}) - r^2 + u_k^2 e^{(u_k - r^2)\nu t}\operatorname{erfc}(u_k\sqrt{\nu t}) = f_3^k(r, t)$$

where u_k are the roots of the polynomial

$$F(r, u) = u^4 + 2r^2u^2 - 4r^3u + r^4 + rg\nu^{-2}$$

We shall use a convolution theorem. Since

$$F_1(s)\,F_2(s) \div \frac{d}{dt}\int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau$$

or

$$\frac{1}{s}F_1(s)\,F_2(s) \div \int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau$$

then from (2.10) and (2.11) with regard for (3.1) we find

$$H = H_0 - \frac{r}{\mu\nu}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\left[\frac{1}{r^4 + r\lambda^3} + \sum_{k=1}^4 \frac{f_3^k(r,\tau)}{u_k F'(u_k)}\right] d\tau =$$

$$= H_0 - \frac{r\rho g H_0}{\mu}\left[-\frac{r}{r^3+\lambda^3} + \sum_{k=1}^4 (u_k^2 - r^2)\frac{f_3^k(r,\tau)}{u_k F'(u_k)}\right] -$$

$$- \frac{r}{\mu}\int_0^t P_0(t-\tau)\left[-\frac{r}{r^3+\lambda^3} + \sum_{k=1}^4 (u_k^2 - r^2)\frac{f_3^k(r,\tau)}{u_k F'(u_k)}\right] d\tau$$

$$Z = \frac{2r^3}{\mu}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\left[\frac{f_1(r,z,\tau)}{r^4 + r\lambda^3} + \sum_{k=1}^4 \frac{f_2^k(r,\tau)}{u_k F'(u_k)}\right] d\tau = \tag{3.2}$$

$$= -\frac{r}{\mu}e^{-r|z|}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\left[\frac{r}{r^3+\lambda^3} + \sum_{k=1}^4 \frac{(u_k^2 - r^2) f_3^k(r,\tau)}{u_k F'(u_k)}\right] d\tau$$

$$Y = -\frac{2i\eta r}{\mu}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\sum_{k=1}^4 \frac{f_2^k(r,z,\tau)\, d\tau}{F'(u_k)} +$$

$$+ \frac{i\eta}{\mu}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\left[\frac{r}{r^3+\lambda^3} + \sum_{k=1}^4 \frac{(u_k^2 - r^2) f_3^k(r,\tau)}{u_k F'(u_k)}\right] d\tau$$

$$P = e^{-r|z|}\frac{d}{dt}\int_0^t [\rho g H_0 + P_0(t-\tau)]\left[-\frac{r^3}{r^3+\lambda^3} + \sum_{k=1}^4 \frac{(u_k^4 - r^4) f_3^k(r,\tau)}{u_k F'(u_k)}\right] d\tau$$

$$(\lambda^3 = g/\nu^2, \quad F'(u_k) = 4(u_k^3 + r^2 u_k - r^3))$$

4. Fourier inversion. Using the inversion formulas for Fourier transforms, we obtain solution

$$v_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X e^{-i(\xi x + \eta y)}\, d\xi d\eta, \qquad v_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y e^{-i(\xi x + \eta y)}\, d\xi d\eta$$
$$v_z = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z e^{-i(\xi x + \eta y)}\, d\xi d\eta, \qquad p = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P e^{-i(\xi x + \eta y)}\, d\xi d\eta \tag{4.1}$$

$$\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H e^{-i(\xi x + \eta y)}\, d\xi d\eta$$

Here the functions X, Y, Z, P and H are determined from Formulas (3.2). The roots of the polinomial $F(r, u)$ which depend on ξ and η appear in these functions. If these roots are known, the solution of the problem under consideration is then obtained in the closed form (4.1). Let us set $u = r\alpha$. Then

$$F(r, u) = r^4(\alpha^4 - 2\alpha^2 - 4\alpha + 1 + \lambda^3 r^{-3}) = r^4 F_1(r, \alpha) \tag{4.2}$$

Examination of the polynomial $F_1(r, \alpha)$ shows that for $r < 1.19813\lambda$ it has two pairs of complex conjugate roots and for $r > 1.19813\lambda$ it has two real roots and one pair of complex conjugate roots. In this connection the integration in (4.1) must be split into the intervals $0 \leqslant r \leqslant 1.19813\,\lambda$ and $r > 1.19813\,\lambda$. Substituting into (4.1) the expression for H from (3.2) and interchanging the order of integration (*) with respect to t and ξ and η, we obtain for the free surface

$$\zeta = \zeta_0 - \frac{1}{2\pi\mu\nu}\frac{d}{dt}\int_0^t\Big\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\rho g H_0 + P_0(t-\tau)]\Big[\frac{1}{r^3+\lambda^3} + \sum_{k=1}^{4} r\frac{f_3^k(r,\tau)}{u_k F'(u_k)}\Big]e^{-i(\xi x+\eta y)}\,d\xi d\eta\Big\}\,d\tau =$$

$$= \zeta_0 - \frac{\rho g}{2\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0\Big[-\frac{r^3}{r^3+\lambda^3} + \sum_{k=1}^{4}\frac{r(u_k^2-r^2)f_3^k(r,t)}{u_k F'(u_k)}\Big]\times$$

$$\times e^{-i(\xi x+\eta y)}d\xi d\eta - \frac{1}{2\pi\mu}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P_0(t-\tau)\Big[-\frac{r^3}{r^3+\lambda^3} + \sum_{k=1}^{4}\frac{r(u_k^2-r^2)f_3^k(r,\tau)}{u_k F'(u_k)}\Big]e^{-i(\xi x+\eta y)}d\xi d\eta d\tau \qquad (4.3)$$

Applying a convolution theorem for Fourier transforms, we find

$$\zeta(x, y, t) = \zeta_0 - \frac{\lambda^3}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_0(x-u,\; y-\omega)\,G_1(u, w, t)\,du\,dw -$$

$$-\frac{1}{2\pi\mu}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_0(x-u,\, y-w,\, t-\tau)\,G_2(u, w, \tau)\,du\,dw\,d\tau$$

after some manipulation

$$\zeta_0 - \frac{1}{2\pi\mu\nu}\frac{d}{dt}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\rho g\zeta_0(x-u,\, y-w) + p_0(x-u,\; y-w,\, t-\tau)]\times$$

$$\times\, G_1(u, w, \tau)\,du\,dw\,d\tau$$

$$G_i = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_i(r, t)\,e^{-i(\xi x+\eta y)}d\xi\,d\eta \qquad (i = 1, 2) \qquad (4.4)$$

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*) For $p = 0$ in (4.3) only the integral with respect to ξ and η remains and the interchange of the order of integration is not required. The validity of such an intergange in the general case will be seen in what follows.

(4.4) cont

$$\psi_1(r,t) = \frac{1}{r^3+\lambda^3} + \sum_{k=1}^{4} [4\alpha_k(\alpha_k^2-1)(\alpha_k^3+\alpha_k-1)r^3]^{-1}[-\alpha_k \operatorname{erf}(r\sqrt{\nu t}) -$$

$$-1+\alpha_k^2 \exp[\nu t r^2(\alpha_k^2-1)]\operatorname{erfc}(-r\alpha_k\sqrt{\nu t})] = (r^3+\lambda^3)^{-1} + \sum_{k=1}^{4}\psi_1^k(r,t)$$

$$\psi_2(r,t) = \sum_{k=1}^{4} \frac{[\alpha_k \exp[\nu t r^2(\alpha_k^2-1)]\operatorname{erfc}(-r\alpha_k\sqrt{\nu t}) - \operatorname{erf}(r\sqrt{\nu t})}{4(\alpha_k^3+\alpha_k-1)} = \sum_{k=1}^{4}\psi_2^k(r,t)$$

Taking the symmetry of the functions ψ_i with respect to the ξ and η axes into account and passing to polar coordinates, we obtain

$$G_i(x,y,t) = \frac{2}{\pi}\int_0^{1/2\pi}\int_0^{\infty}\psi_i(r,t)\cos(Rr\cos\theta)\,dr\,d\theta$$

$$R^2 = x^2+y^2, \qquad \theta = (R, r) \qquad (i=1,2) \tag{4.5}$$

Depending on the specific problem one or another form of the formulas (4.4) can be applied.

In order to avoid determining the roots α_k of the polynomial $F_1(r,\alpha)$, it is expedient in the calculation of G_i to make the following substitution in the integrand:

$$r = r(\alpha_k) = -\lambda(\alpha_k^4 + 2\alpha_k^2 - 4\alpha_k + 1)^{-1/3} \tag{4.6}$$

After the indicated substitution, we have

$$G_i(x,y,t) = \frac{2}{\pi}\int_0^{1/2\pi}\sum_{k=1}^{4}\int_{L_k}\psi_i^k(r,t)\cos[Rr(\alpha_k)\cos\theta]\,d\alpha_k\,d\theta +$$

$$+\frac{2}{\pi}\gamma_i\int_0^{1/2\pi}\int_0^{\infty}\frac{\cos(Rr\cos\theta)\,dr\,d\theta}{r^3+\lambda^3}, \quad \gamma_i = \begin{cases}0 & \text{for } i=2\\ 1 & \text{for } i=1\end{cases} \tag{4.7}$$

The branch of the curve, corresponding to the root α_k, along which the integration must be carried out, is denoted by L_k.

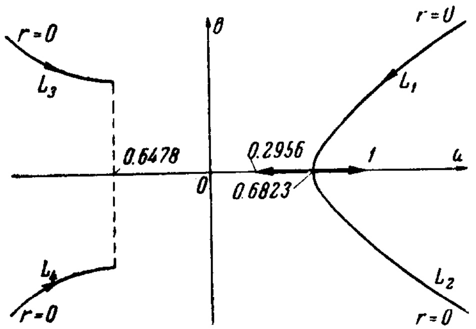

Fig. 1

Substituting $\alpha = a + ib$ in the polynomial $F_1(r,\alpha)$ and equating the real and imaginary parts to zero, we obtain

$$b = 0$$

$$a^4 + 2a^2 - 4a + 1 + \lambda^3 r^{-3} = 0 \tag{4.8}$$

$$b = \pm(a^2+1-a^{-1})^{1/2}$$

$$4a^6 + 4a^4 - 1 + \lambda^3 a^2 r^{-3} = 0 \tag{4.9}$$

As previously indicated, the second equation of (4.8) gives only two roots for a when $r \geqslant 1.19813\lambda$. The first equation of the system (4.9) gives a relation between the real and imaginary parts of the roots, and the second gives a relation between the real part and the parameter. An exemplary graph of the branches L_k is presented in the figure; the direction of the path of the branches L_k as r varies from $-\infty$ to $+\infty$ is shown by arrows.

Developing the sum in (4.7), we obtain

$$G_i(x, y, t) = \frac{2}{\pi}\Big\{\int_0^{1/2\pi}\int_{0.6823}^{1} \varphi_i(\alpha)\cos[Rr(\alpha)\cos\theta]\,d\alpha\,d\theta - \\ - \int_0^{1/2\pi}\int_{0.2956}^{0.6823} \varphi_i(\alpha)\cos[Rr(\alpha)\cos\theta]\,d\alpha\,d\theta + \\ + \int_0^{1/2\pi}\int_{L_1} \varphi_i(\alpha_1)\cos[Rr(\alpha_1)\cos\theta]\,d\alpha_1\,d\theta + \\ + \int_0^{1/2\pi}\int_{L_2} \varphi_i(\alpha_2)\cos[Rr(\alpha_2)\cos\theta]\,d\alpha_2\,d\theta + \\ + \int_0^{1/2\pi}\int_{L_3} \varphi_i(\alpha_3)\cos[Rr(\alpha_3)\cos\theta]\,d\alpha_3\,d\theta + \\ + \int_0^{1/2\pi}\int_{L_4} \varphi_i(\alpha_4)\cos[Rr(\alpha_4)\cos\theta]\,d\alpha_4\,d\theta + \int_0^{1/2\pi}\int_0^{\infty}\frac{\gamma_i\cos(Rr\cos\theta)}{r^3+\lambda^3}\,dr\,d\theta\Big\} \tag{4.10}$$

$$\varphi_1(\alpha) = \frac{r^2(\alpha)}{3\alpha(\alpha^2-1)\lambda^3}\left[-\alpha\,\mathrm{erf}\,(r\sqrt{\nu t}) - 1 + \alpha^2 e^{\nu t r^2(\alpha^2-1)}\,\mathrm{erfc}\,(r\alpha\sqrt{\nu t})\right]$$

$$\varphi_2(\alpha) = \frac{r^4(\alpha)}{3\alpha\lambda^3}\left[\alpha^2 e^{\nu t r^2(\alpha^2-1)}\,\mathrm{erfc}\,(-r\alpha\sqrt{\nu t}) - \alpha\,\mathrm{erf}\,(r\sqrt{\nu t})\right], \quad \gamma_i = \begin{cases}1 & \text{for } i=1\\ 0 & \text{for } i=2\end{cases}$$

In order to reduce the integration in (4.10) to the real dimain, we shall use the relation (4.9) between the real and imaginary parts of the roots and the relation between r and the real part of the roots a. We have

$$b = (a^2 + 1 - a^{-1})^{1/2}, \qquad r(a \pm ib) = h(a) = \lambda a^{3/2}(4a^6 + 4a^4 - 1)^{-1/2}$$

$$d\alpha = d(a \pm ib) = da \pm i\,db = \left(1 \pm \frac{2a + a^{-2}}{2b}\right)da \tag{4.11}$$

Taking into account that α_3, α_4 and α_2, α_1 are complex conjugate roots, we reduce $G_i(x, y, t)$ to the form

$$G_i(x, y, t) = \frac{2}{\pi}\int_0^{1/2\pi}\int_{0.6823}^{1}\varphi_i(\alpha)\cos[Rr(\alpha)\cos\theta]\,d\alpha\,d\theta - \\ - \frac{2}{\pi}\int_0^{1/2\pi}\int_{0.2956}^{0.6823}\varphi_i(\alpha)\cos[Rr(\alpha)\cos\theta]\,d\alpha\,d\theta - \\ - \frac{1}{\pi}\int_0^{1/2\pi}\int_{0.6823}^{\infty}[\varphi_i(a+ib)+\varphi_i(a-ib)]\cos[Rh(a)\cos\theta]\,da\,d\theta + \tag{4.12}$$

$$+\frac{1}{\pi}\int_0^{1/2\pi}\int_{-\infty}^{-0.6478}[\varphi_i(a+ib)+\varphi_i(a-ib)]\cos[Rh(a)\cos\theta]\,da\,d\theta- \qquad \text{(cont.) (4.12)}$$

$$-\frac{i}{\pi}\int_0^{1/2\pi}\int_{0.6823}^{\infty}[\varphi_i(a+ib)-\varphi_i(a-ib)]\left(2a+\frac{1}{a^2}\right)\frac{\cos[Rh(a)\cos\theta]}{2b}\,da\,d\theta+$$

$$+\frac{i}{\pi}\int_0^{1/2\pi}\int_{-\infty}^{-0.6478}[\varphi_i(a+ib)-\varphi_i(a-ib)]\left(2a+\frac{1}{a^2}\right)\frac{\cos[Rh(a)\cos\theta]}{2b}\,da\,d\theta+$$

$$+\frac{2\gamma_i}{\pi}\int_0^{1/2\pi}\int_0^{\infty}\frac{\cos[Rr\cos\theta]}{r^3+\lambda^3}\,dr\,d\theta \qquad (i=1,2), \qquad \gamma_i=\begin{cases}1 & \text{for } i=1\\ 0 & \text{for } i=2\end{cases}$$

It is easily shown that

$$\operatorname{Re}\{e^{((a+ib)^2-1)h^2(a)\nu\tau}\operatorname{erfc}[-h(a)(a+ib)\sqrt{\nu\tau}]\}=H_1e^{-h^2(a)\nu\tau}$$

$$H_1=2\pi^{-1/2}\int_0^{\infty}e^{-x^2+2ah(a)x\sqrt{\nu\tau}}\cos[2bh(a)x\sqrt{\nu\tau}]\,dx$$

$$\operatorname{Im}\{e^{((a+ib)^2-1)h^2(a)\nu\tau}\operatorname{erfc}[-h(a)(a+ib)\sqrt{\nu\tau}]\}=H_2e^{h^2(a)\nu\tau}$$

$$H_2=2\pi^{-1/2}\int_0^{\infty}e^{-x^2+2ah(a)x\sqrt{\nu\tau}}\sin[2bh(a)x\sqrt{\nu\tau}]\,dx$$

The determination of the integrand functions does not present any particular difficulty

$$\varphi_1(a+ib)+\varphi_1(a-ib)=\frac{2h^2(a)}{3\lambda^3(a^4+b^4+1+2a^2b^2-2a^2+2b^2}\times$$
$$\times\Big\{-(a^2-b^2-1)\operatorname{erf}(h(a)\sqrt{\nu\tau})-\frac{a(a^2-3b^2-1)}{a^2+b^2}+$$
$$+H_1a(a^2-3b^2-1)e^{-\nu\tau h^2(a)}-H_2b(b^2-3a^2+1)e^{-\nu\tau h^2(a)}\Big\}$$

$$i[\varphi_1(a+ib)-\varphi_1(a-ib)]=\frac{2h^2(a)}{3\lambda^3(a^4+b^4+1+2a^2b^2-2a^2+2b^2)}\times$$
$$\times\Big\{2ab\operatorname{erf}(h(a)\sqrt{\nu\tau})-\frac{b(b^2-3a^2+1)}{a^2+b^2}-$$
$$-H_1b(b^2-3a^2+1)e^{-\nu\tau h^2(a)}-H_2a(a^2-3b^2+1)e^{-\nu\tau h^2(a)}\Big\}$$

$$\varphi_2(a+ib)+\varphi_2(a-ib)=\frac{2h^4(a)}{3\lambda^3}[-\operatorname{erf}(h(a)\sqrt{\nu\tau})+H_1ae^{-h^2(a)\nu\tau}-$$
$$-H_2be^{-h^2(a)\nu\tau}]$$

$$i[\varphi_2(a+ib)-\varphi_2(a-ib)]=-\frac{2h^4(a)}{3\lambda^3}[H_2ae^{-h^2(a)\nu\tau}+H_1be^{-h^2(a)\nu\tau}]$$

The integrals H_1 and H_2 are not obtained in elementary functions. We shall calculate them approximately, using the method of Laplace [4].

It is now easily established that the integrand in (4.3) is continuous and the integral with respect to r is absolutely convergent. Consequently, the interchange of the order of integration in (4.3) is valid.

5. Calculation of integrals and computation formulas. Let us consider the integrals

$$f_1(a_1, b_1) = \int_0^\infty \exp(-x^2 + 2a_1x)\cos 2b_1 x\,dx$$
$$f_2(a_1, b_1) = \int_0^\infty \exp(-x^2 + 2a_1x)\sin 2b_1 x\,dx \tag{5.1}$$

Applying the method of Laplace to the calculation of these integrals and requiring that their approximate values coincide with the exact ones for $a_1 = 0$ and $b_1 = 0$, we can write

$$f_1(a_1, b_1) \approx {}^1/_2 \sqrt{\pi}\exp(a_1^2 - b_1^2)\,[\operatorname{erfc}(-a_1)\cos 2a_1b_1 - \operatorname{erf}(b_1)\sin 2a_1b_1]$$
$$f_2(a_1, b_1) \approx {}^1/_2 \sqrt{\pi}\exp(a_1^2 - b_1^2)\,[\operatorname{erfc}(-a_1)\sin 2a_1b_1 + \operatorname{erf}(b_1)\cos 2a_1b_1]$$

Setting $a_1 = ah(a)\sqrt{\nu\tau}$, $b_1 = bh(a)\sqrt{\nu\tau}$, here, we obtain

$$H_1 \approx \exp[h^2(a)(a^2 - b^2)\nu\tau]\,\{\operatorname{erfc}[-ah(a)\sqrt{\nu\tau}]\cos 2abh^2(a)\nu\tau - \operatorname{erf}[bh(a)\sqrt{\nu\tau}]\sin 2abh^2(a)\nu\tau\}$$
$$H_2 \approx \exp[h^2(a)(a^2 - b^2)\nu\tau]\,\{\operatorname{erfc}[-ah(a)\sqrt{\nu\tau}]\sin 2abh^2(a)\nu\tau + \operatorname{erf}[bh(a)\sqrt{\nu\tau}]\cos 2\,abh^2(a)\nu\tau\} \tag{5.2}$$

Let us now proceed to the calculation of the integrals which appear in (4.12), applying the method of stationary phase.

It is easily established that the asymptotic value of the integral differences with respect to the real roots is equal to zero. Integrals which contain a trigonometric part of the form $\cos[Rh(a)\cos\theta]$, are also asymptotically equal to zero. Consideration of the integrals which contain factors of the form

$$\cos[Rh(a)\cos\theta]\sin 2abh^2(a)\nu\tau = {}^1/_2\{\sin[Rh(a)\cos\theta - 2abh^2(a)(\nu\tau] + \sin[Rh(a)\cos\theta + 2abh^2(a)\nu\tau]\}$$

$$\cos[Rh(a)\cos\theta]\cos 2abh^2(a)\nu\tau = {}^1/_2\{\cos[Rh(a)\cos\theta - 2abh^2(a)\nu\tau] + \cos[Rh(a)\cos\theta + 2abh^2(a)\nu\tau]\}$$

lead to equations for determining the stationary points

$$-(8a^6 + 4a^4 + 1)(4a^6 + 4a^4 - 1)^{1/3}(a^3 + a - 1)^{1/2} \mp \mp a^{7/6}\omega\,(16a^9 + 24a^7 - 52a^6 + 8a^5 - 20a^4 + 20a^3 + 14a - 11) = 0$$
$$-(8a^6 + 4a^4 + 1)(4a^6 + 4a^4 - 1)^{1/3}(a^3 + a - 1)^{1/2} \mp \mp a^{7/6}\omega\,(16a^9 + 24a^7 + 52a^6 + 8a^5 + 20a^4 + 20a^3 + 14a + 11) = 0 \tag{5.3}$$

$$(\omega = \lambda\nu\tau\,(2R\cos\theta)^{-1}) \tag{5.4}$$

To find the exact relation of a to ω is very difficult, but it is possible to obtain approximate formulas for small and large values of ω

If the value of $\omega < \frac{1}{2}$, it can be regarded as small; if the value of $\omega > 2$, it can be regarded as large.

1. For small values of ω and $b = (a^2 + 1 - a^{-1})^{\frac{1}{2}}$

$$[h(a)+4abh^2(a)\omega]\,\lambda R\cos\theta \to a \approx 0.6823+0.065\,\omega^2$$
$$[h(a)-4abh^2(a)\omega]\,\lambda R\cos\theta \to a \approx {}^1/_2\sqrt{2\omega}^{\,-3/2} \tag{5.5}$$

2. For small values of ω and $b_1 = (a^2+1+a^{-1})^{\frac{1}{2}}$

$$[h(a)+4abh^2(a)\omega]\,\lambda R\cos\theta \to \text{there are no stationary points}$$

$$[h(a)-4abh^2(a)\omega]\,\lambda R\cos\theta \to \begin{cases} a \approx 0.6478+20\omega^3 & (a<1.8) \\ a \approx {}^1/_2\sqrt{2\omega}^{\,-3/2} & (a>1.8)\end{cases} \tag{5.6}$$

3. For large values of ω and $b = (a^2+1-a^{-1})^{\frac{1}{2}}$

$$[h(a)+4abh^2(a)\omega]\,\lambda R\cos\theta \to a \approx 1.02367-0.57\omega^{-1}$$
$$[h(a)-4abh^2(a)\omega]\,\lambda R\cos\theta \to a \approx 1.02367+0.57\omega^{-1} \tag{5.7}$$

For large values of ω and $b_1 = (a^2+1+a^{-1})^{\frac{1}{2}}$ there are no ststionary points. It is possible to show that for $a < 1.8$ the integrals in (4.12) can be neglected.

Let us consider the case of small values of ω $(\omega < \frac{1}{2})$. The method of stationary phase gives the following formula

$$\int_\alpha^\beta g(t)\,e^{ikh(t)}\,dt = \left[\frac{2\pi}{|kh''(\tau)|}\right]^{1/2} g(\tau)\exp\left(ikh(\tau)\pm i\frac{\pi}{4}\right)\left[1+O\left(\frac{1}{\sqrt{k}}\right)\right] \tag{5.8}$$

where τ is a ststionary point of the function $h(t)$ $(\alpha<\tau<\beta)$, and k is a large parameter. The sign of the exponential term $(\pm i\pi/4)$ is taken to be the same as the sign of $kh''(\tau)$.

After some calculations we obtain from (4.10)

$$G_1 = -\int_0^{1/2\pi} \frac{\sqrt{g}\nu^2\tau^3}{8R^3\cos^3\theta\sqrt{\pi R\cos\theta}}\exp\left(-\frac{\nu\tau^5g^2}{8R^4\cos^4\theta}\right)\cos\left[\frac{\pi}{4}-\frac{g\tau^2}{4R\cos\theta}\right]\times$$
$$\times\left[1+O\left(\frac{1}{\sqrt{\lambda R\cos\theta}}\right)\right]d\theta$$

$$G_2 = \int_0^{1/2\pi} \frac{\sqrt{g}\,g\nu\tau^4}{16(R\cos\theta)^{9/2}\sqrt{\pi}}\exp\left(-\frac{\nu\tau^5g^2}{8R^4\cos^4\theta}\right)\sin\left[\frac{\pi}{4}-\frac{g\tau^2}{4R\cos\theta}\right]\times$$
$$\times\left[1+O\left(\frac{1}{\sqrt{\lambda R\cos\theta}}\right)\right]d\theta\ \ (\lambda r\to\infty) \tag{5.9}$$

To calculate the integral with respect to the variable θ we again apply the method of stationary phase. In this case $\theta = 0$, from whence it follows

$$G_1 = -\frac{\nu^2\tau^2}{8\sqrt{2}R^3}\exp\left[-\frac{\nu g^2\tau^5}{8R^4}\right]\cos\left(\frac{g\tau^2}{4R}\right)\left[1+O\left(\frac{1}{\sqrt{\lambda R}}\right)+O\left(\sqrt{\frac{4R}{gt^2}}\right)\right]$$
$$G_2 = \frac{g\nu\tau^3}{4\sqrt{2}R^4}\exp\left[-\frac{\nu g^2\tau^5}{8R^4}\right]\sin\left(\frac{g\tau^2}{4R}\right)\left[1+O\left(\frac{1}{\sqrt{\lambda R}}\right)+O\left(\sqrt{\frac{4R}{gt^2}}\right)\right]$$
$$(\lambda R\to\infty,\ gt^2/4R\to\infty) \tag{5.10}$$

We finally obtain

$$\zeta(x, y, t) = \zeta_0(x, y) - \frac{1}{16\sqrt{2}\rho\pi}\frac{d}{dt}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\rho g\zeta_0(x-u, y-w) + p_0(x-u, y-w, t-\tau)]\, G_1 du\, dw\, d\tau =$$

$$= \zeta_0(x, y) + \frac{g}{16\sqrt{2}\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_0(x-u, y-w)\, G_1(u, w, t)\, du\, dw -$$
$$- \frac{g}{8\sqrt{2}\rho\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(x-u, y-w, t-\tau)\, G_2(u, w, \tau)\, du\, dw\, d\nu \qquad (5.11)$$

In the case of axial symmetry Formula (5.11) takes the form

$$\zeta(r, t) \sim \zeta_0(r) + \frac{g}{16\sqrt{2}\pi}\int_0^{2\pi}\int_0^{\infty}\zeta_0(R)\, t^2 R_1^{-3}\exp\left(-\frac{\nu t^5 g^2}{8R_1^4}\right)\cos\frac{gt^2}{4R_1} R\, dR\, d\varphi -$$
$$- \frac{g}{8\sqrt{2}\rho\pi}\int_0^t\int_0^{\infty}\int_0^{2\pi} p_0(R, t-\tau)\,\tau^3 R_1^{-4}\exp\left(-\frac{\nu\tau^5 g^2}{8R_1^4}\right)\sin\frac{g\tau^2}{4R_1} R\, dR\, d\varphi\, d\tau =$$
$$= \zeta_0(r) - \frac{1}{16\sqrt{2}\rho\pi}\frac{d}{dt}\int_0^t\int_0^{2\pi}\int_0^{\infty}[\rho g\zeta_0(R) + p_0(R, t-\tau)]\,\tau^2 R_1^{-3} \times$$
$$\times \exp\left(-\frac{\nu g^2\tau^5}{8R_1^4}\right)\cos\frac{g\tau^2}{4R_1} R\, dR\, d\varphi\, d\tau \qquad (5.12)$$

$$R_1 = R^2 + r^2 + 2rR\cos\varphi \quad (\lambda R \to \infty,\ gt^2/4R \to \infty)$$

Since the magnitude of $\lambda = (g\nu^{-2})^{1/3}$ is large for water, Formulas (5.11) and (5.12) are suitable practically for all values of r different from zero.

Formulas (5.11) and (5.12) give an asymptotic expression for determining the form of the free surface for small values of ω $(\omega < \frac{1}{2})$. Further solution is possible only after prescribing the form of the functions $\zeta_0(x, y)$ and $p_0(x, y, t)$.

6. Particular cases. For the mechanical integration of the obtained results, let us consider some particular cases.

1. Set

$$p_0 \equiv 0, \qquad \zeta_0 = A(2\pi r)^{-1}\delta(r), \qquad r^2 = x^2 + y^2$$

Prescribing such conditions on the free surface corresponds to an elevation of a finite volume of fluid in a small area near the origin of the coordinate system. Since $r^{-1}\delta(r) \sim 0$, we then obtain from (5.12)

$$\zeta(r, t) = A\frac{gt^2}{16\sqrt{2}r^3}\exp\left(-\frac{\nu g^2 t^5}{8r^4}\right)\cos\frac{gt^2}{4r}\left[1 + O\left(\frac{1}{\sqrt{\lambda r}}\right) + O\left(\sqrt{\frac{4r}{gt^2}}\right)\right] \qquad (6.1)$$
$$(\lambda r \to \infty,\ gt^2(4r)^{-1} \to \infty)$$

Here the formula

$$\int_0^{\infty} f(r)\,\delta(r)\,dr = f(0)$$

has been used.

Let us assume that the oscillation of the level is observed at a given place. Then it follows from Formula (6.2) that due to the presence of the factor t^2 the amplitude of the oscillation at first increases and the period of the oscillation decreases. But due to the presence of the exponential factor the amplitude of the oscillation begins to tend to zero with time; at the place under consideration an unlimited reduction of the amplitudes and periods of oscillation are observed. This characteristic does not occur in the Cauchy-poisson problem for an ideal fluid. The free surface of the viscous fluid in this case is a series of annular waves propagating with the velocity $c = 2rt^{-1}$.

2. Set

$$\zeta_0 \equiv 0, \qquad p_0 = Q\delta(x)\,\delta(y)\,\delta(t)$$

This denotes that at the initial moment of time a pressure impulse acts on the surface of the fluid in a small area near the origin of the coordinate system. From Formula (5.11) we obtain

$$\zeta = -\frac{Qgt^3}{8\sqrt{2}} \exp\left(-\frac{\nu g^2 t^5}{8r^4}\right) \sin\frac{gt^2}{4r}\left[1 + O\left(\frac{1}{\sqrt{\lambda r}}\right) + O\left(\sqrt{\frac{4r}{gt^2}}\right)\right]$$

$$(\lambda r \to \infty,\ gt^2(4r)^{-1} \to \infty) \tag{6.2}$$

From the expression obtained it is seen that the picture of the free surface in general coincides with the corresponding picture for an ideal fluid. The difference consists in that for $r \to 0$ there is no unlimited growth of amplitude, which occurs in the case of the ideal fluid. For $\nu = 0$ we obtain the well-known result for an ideal fluid [5]. Formula (6.2) permits a solution to the problem of ship waves on the surface of a viscous fluid of infinite depth for straight and curved ship course lines.

3. Set

$$\zeta_0 \equiv 0, \qquad p_0(x, y, t) = Q\delta(x)\,\delta(y)\,(1 - \cos\sigma t)$$

It can be considered that an oscillator is acting on the fluid in a small area near the origin of the coordinate system. Using Formula (5.11) and applying the method of stationary phase, we obtain

$$\zeta = \frac{Q\sigma^3}{\rho g^2\sqrt{2\pi gr}} \exp\left(-\frac{8\sigma^5\nu r}{g^3}\right) \sin\left[\frac{\pi}{4} - \frac{\sigma^2 r}{g} + \sigma t\right]\left[1 + O\left(\frac{1}{\sqrt{\lambda r}}\right) + O\left(\sqrt{\frac{4r}{gt^2}}\right) + O\left(\sqrt{\frac{g}{\sigma^2 r}}\right)\right] \tag{6.3}$$

$$(\lambda r \to \infty,\ (4r)^{-1} gt^2 \to \infty,\ g(\sigma^2 r)^{-1} \to \infty)$$

The free surface represents progressive waves whose amplitudes damp with distance from the origin of the coordinate system and which are propagating with the velocity $c = g\sigma^{-1}$.

In all cases we obtain the well-known results for an ideal fluid at $\nu = 0$.

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